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FRACTIONAL INTEGRALS ON WEIGHTED H^p SPACES

BY

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ABSTRACT. We characterize the pairs of doubling weights (u,v) on R^n such that

$$\|I_\alpha f\|_{H^q_u} \le c\|f\|_{H^p_v}, \qquad 0 < p < q < \infty,$$

where I_α, $\alpha > 0$, is the fractional integral operator. We also consider the behavior of an associated maximal function. Applications of the results to Sobolev inequalities in weighted L^p spaces are given.

1. Introduction and notation. In this paper we will characterize the pairs of doubling weights (u,v) on R^n such that

$$\|I_\alpha f\|_{H^q_u} \le c\|f\|_{H^p_v}, \qquad 0 < p < q < \infty,$$

where I_α, $\alpha > 0$, is the fractional integral operator. We shall also give applications of these results to Sobolev inequalities in weighted L^p spaces.

A weight function u is said to belong to D_μ, $\mu \ge 1$, if $u(tQ) \le ct^{n\mu}u(Q)$ for every $t \ge 1$ and every cube $Q \subset R^n$, where $u(Q)$ denotes the u-measure of Q and tQ is the cube with the same center as Q but with t times the edgelength. We write $D_\infty = \bigcup_{\mu \ge 1} D_\mu$. Analogously, $u \in RD_\nu$, $\nu > 0$, if $u(tQ) \ge ct^{n\nu}u(Q)$ for every $t \ge 1$ and every cube Q. It is not hard to see that if $u \in D_\infty$, then $u \in RD_\nu$ for some $\nu > 0$. We write $u \in RH_r$, $r > 1$, if

$$\left(\frac{1}{|Q|}\int_Q u(x)^r\,dx\right)^{1/r} \le c\frac{1}{|Q|}\int_Q u(x)\,dx \quad \text{for all } Q.$$

$\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}_{0,0}$ will denote respectively the class of Schwartz functions, the tempered distributions and the class of functions in $\mathcal{S}$ whose Fourier transform has compact support disjoint from the origin. L^p_u is the class of functions g such that $\|g\|_{L^p_u} = (\int |g(x)|^p u(x)\,dx)^{1/p}$ is finite. If $\phi \in \mathcal{S}$, $\int \phi(x)\,dx \neq 0$ and $f \in \mathcal{S}'$, then we denote as usual $F(x,t) = (f * \phi_t)(x)$, where $\phi_t(x) = t^{-n}\phi(x/t)$, and

$$N(f)(x) = \sup_{(y,t)\in\Gamma_a(x)} |F(y,t)|, \quad \text{with } \Gamma_a(x) = \{(y,t)\colon y \in R^n,\ t > 0,\ |x-y| \le at\},$$
$$a > 0.$$

The space H^p_u, $0 < p < \infty$, consists of all distributions $f \in \mathcal{S}'$ such that $\|f\|_{H^p_u} = \|N(f)\|_{L^p_u}$ is finite. If $u \in D_\infty$, the definition of H^p_u is independent of ϕ and a, and $\|f\|_{H^p_u}$ is equivalent to $\|N_0(f)\|_{L^p_u}$, where $N_0(f)(x) = \sup_{t>0}|F(x,t)|$ (see [**8**]).

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If $\alpha \geq n$, we write

$$N_\alpha = \begin{cases} \alpha - n & \text{if } \alpha \text{ is an integer,} \\ [\alpha - n - 1] + 1 & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the integer part of x, and define $\mathcal{S}_\alpha$ for $0 < \alpha < n$ by $\mathcal{S}_\alpha = \mathcal{S}$ and for $\alpha \geq n$ by $\mathcal{S}_\alpha = \{f \in \mathcal{S} \colon \int f(y) y^\beta \, dy = 0,\ |\beta| \leq N_\alpha\}$. $I_\alpha f$ is defined by means of its Fourier transform, $(I_\alpha f)\hat{}(x) = |x|^{-\alpha} \hat{f}(x)$, and it can be shown (see [**6**]) that for $f \in \mathcal{S}_\alpha$, $I_\alpha f(x) = (k_\alpha * f)(x)$, where

$$k_\alpha(x) = \begin{cases} c_{\alpha,n} |x|^{-n+\alpha} & \text{if } \alpha \neq n + 2m, \\ c_{\alpha,n} |x|^{-n+\alpha} \log |x| & \text{if } \alpha = n + 2m, \end{cases}$$

$m = 0, 1, 2, \ldots$. Also, by checking Fourier transforms, we have

$$I_\alpha f(x) = c \int_0^\infty t^{\alpha - 1} F(x,t) \, dt$$

if $f \in \mathcal{S}_\alpha$ and the convolver ϕ is radial (see (3.2)).

For $\alpha, \varepsilon \in R$, u a nonnegative function and $f \in \mathcal{S}'$, we introduce the following maximal function:

$$M_{\alpha,u,\varepsilon}(f)(x) = \sup_{t>0} t^\alpha u(B_t(x))^{\varepsilon/n} |F(x,t)|,$$

where $B_t(x)$ denotes the ball with center x and radius t.

For this maximal operator we will characterize the pairs of weights (u, v) for which H_v^p is mapped into L_u^q. In proving this, we will use Carleson measures. We then obtain results for fractional integrals by proving weighted versions of inequalities due to Welland [**11**] and Hedberg [**7**].

Statement of the results.

THEOREM 1. *Let $0 < p \leq q < \infty$, $u \in D_\mu \cap RD_\nu$, $v \in D_\infty$ and $\alpha, \varepsilon \in R$ satisfy $\varepsilon > -\alpha/\mu$ if $\alpha > 0$ and $\varepsilon > -\alpha/\nu$ if $\alpha \leq 0$. Then*

$$\|M_{\alpha,u,\varepsilon}(f)\|_{L_u^q} \leq c \|f\|_{H_v^p}$$

if and only if

$$|Q|^{\alpha/n} u(Q)^{\varepsilon/n + 1/q} \leq c v(Q)^{1/p} \quad \textit{for every cube } Q. \tag{1.1}$$

In case $\varepsilon = 0$, we will see that the sufficiency statement in Theorem 1 is true without the assumption that $u \in D_\infty$. This case is especially interesting since for nonnegative functions f and ϕ it is not hard to see that

$$\sup_{t>0} t^\alpha F(x,t) \approx \sup_{Q \colon x \in Q} \frac{1}{|Q|^{1-(\alpha/n)}} \int_Q f(y) \, dy.$$

Moreover, the sufficiency statement can be modified as follows for $\varepsilon \geq 0$ without assuming $u \in D_\infty$: if $0 < p \leq q < \infty$, $v \in D_\infty \cap RD_\nu$, $\varepsilon \geq 0$ and $\alpha > -\nu\varepsilon q/p$, then (1.1) implies that $\|M_{\alpha,u,\varepsilon}(f)\|_{L_u^q} \leq c \|f\|_{H_v^p}$.

THEOREM 2. *Let $\alpha > 0$, $0 < p < q < \infty$, $u, v \in D_\infty$. Then*

$$\|I_\alpha f\|_{H_u^q} \leq c \|f\|_{H_v^p} \quad \textit{for every } f \in \mathcal{S}_\alpha$$

if and only if

$$|Q|^{\alpha/n} u(Q)^{1/q} \leq c v(Q)^{1/p} \quad \textit{for every cube } Q. \tag{1.2}$$

We will also prove the following theorem in which it is not assumed that $u \in D_\infty$.

THEOREM 3. *Let $\alpha > 0$, $0 < p < n/\alpha$, $1/p^* = 1/p - \alpha/n$ and $p < q$. Then if $v \in RH_{p^*/p}$ and the pair (u, v) satisfies* (1.2) *we have*

$$\|I_\alpha f\|_{L^q_u} \le c\|f\|_{H^p_v} \quad \textit{for every } f \in S_\alpha.$$

If also $p > 1$ and v has the property that $\|f\|_{H^p_v} \approx \|f\|_{L^p_v}$ for $f \in S_\alpha$, then we may replace $\|\cdot\|_{H^p_v}$ by $\|\cdot\|_{L^p_v}$ in the theorems above. This is certainly the case if $v \in A_p$, i.e., if for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1} \le c,$$

and is even true for more general v (see [**2**] and [**9**]). Note also that $\|I_\alpha f\|_{L^q_u} \le \|I_\alpha f\|_{H^q_u}$ for $f \in S_\alpha$, so that our theorems also yield results for L^q_u and L^p_v.

The technique used to prove Theorem 2 can be used to obtain the following result of Strömberg and Wheeden [**10**], which concerns the case $p = q$ in Theorem 2.

THEOREM 4. *Let $\alpha > 0$, $0 < p < \infty$, $u \in D_\infty$, $\{a_k\}_{k=1}^m$ be a sequence of distinct points in R^n and*

$$\Pi_\alpha(x) = (1 + |x|)^\alpha \prod_{k=1}^m \left(\frac{|x - a_k|}{1 + |x - a_k|}\right)^{\alpha_k}, \qquad 0 \le \alpha_k \le \alpha.$$

Then, if $v(x) = \Pi_\alpha(x)^p u(x)$, we have

$$\|I_\alpha f\|_{H^p_u} \le c\|f\|_{H^p_v} \quad \textit{for every } f \in S_\alpha.$$

In case $0 < p \le 1$, Theorem 2 was proved by a different technique in [**10**]. In case $p > 1$, Theorem 2 is an improvement of a result in [**10**] where it is assumed that u satisfies the stronger condition $u \in A_\infty$, where A_∞ denotes $\bigcup_{p>1} A_p$.

Theorem 4 has an analogue for $q > p$. In fact, it is shown in [**10**] that if $0 < p < \infty$, $0 < 1/p - 1/q \le \alpha/n$, $\beta/n = \alpha/n - (1/p - 1/q)$ and

$$\Pi_\beta(x) = (1 + |x|)^\beta \prod_{k=1}^m \left(\frac{|x - a_k|}{1 + |x - a_k|}\right)^{\beta_k}, \qquad 0 \le \beta_k \le \beta,$$

then for $u \in A_\infty$ and $v(x) = u(x)^{p/q}\Pi_\beta(x)^p$, we have

$$\|I_\alpha f\|_{H^q_u} \le c\|f\|_{H^p_v} \quad \text{for every } f \in S_\alpha.$$

This can be obtained as a corollary of Theorem 2 by using the simple fact from [**10**] that the pair (u, v) satisfies (1.2).

As an application of Theorems 2 and 4, we shall deduce in §4 the following Sobolev type inequality:

If $1 < p \le q < \infty$, then

$$\|f\|_{L^q_{|x|^{q\gamma}}} \le c\|\nabla f\|_{L^p_{|x|^{p\beta}}} \quad \text{for every } f \in C_0^\infty \tag{1.3}$$

with support disjoint from the origin if and only if

$$1/q + \gamma/n = 1/p + (\beta - 1)/n \ne 0, \tag{1.4}$$

$$1/p - 1/q \le 1/n. \tag{1.5}$$

When $1/q > -\gamma/n$, (1.3) is valid for every $f \in C_0^\infty$ by passing to the limit. This last case is contained in the results given by Caffarelli, Kohn and Nirenberg in [**3**]. The case $1/q < -\gamma/n$ can be obtained from the case $1/q > -\gamma/n$, and vice-versa, by changing variables. Actually, as we shall see in §4, if (1.3) holds for a given p and q and a single pair γ, β satisfying (1.4), then it holds for the same p and q and all pairs γ, β satisfying (1.4), the class of f being those C_0^∞ functions with support disjoint from 0.

The exclusion in (1.4) of the case $1/q + \gamma/n = 0$ is necessary since the inequality $\|f\|_{L^q_{|x|^{-n}}} \le c\|\nabla f\|_{L^p_{|x|^{p-n}}}$ does not hold for all $f \in C_0^\infty$ with support disjoint from 0. This can be seen by choosing $f(x) = \phi(|x|)$, where given $0 < \varepsilon < R < \infty$, $\phi(t) = 1$ if $\varepsilon \le t \le R$, $\phi(t) = 0$ if $t < \varepsilon/2$ or if $t > 2R$, and ϕ is essentially linear elsewhere. In fact, the left side is then equivalent to $[\log(R/\varepsilon)]^{1/q}$, while the right is bounded by a constant independent of ε and R.

2. Proof of Theorem 1. We begin by recalling a well-known result on Carleson measures (for a simple proof, see Theorem 2 of [**5**]). We shall use the notation $B_Q = \{(x,t) \colon x \in Q,\ 0 < t \le l\}$, where Q is a cube and l is the length of its side.

LEMMA (2.1). *Let σ and v be nonnegative measurable functions on R_+^{n+1} and R^n respectively, and $v \in D_\infty$. Then for $0 < p \le q < \infty$,*

$$\left(\int_{R_+^{n+1}} |F(x,t)|^q\, \sigma(x,t)\, dx\, dt\right)^{1/q} \le c_1 \left(\int_{R^n} N(f)(x)^p v(x)\, dx\right)^{1/p}$$

for every measurable f on R^n if and only if $\sigma(B_Q) \le c_2 v(Q)^{q/p}$ for every cube Q. The relationship between c_1 and c_2 is given by $c_1 = c_{p,q} c_2^{1/q}$, where $c_{p,q}$ is a constant depending only on p and q.

Also, we will use the following mean value inequality due to Strömberg and Torchinsky [**8**]: Let $\phi \in \mathcal{S}$, $\int \phi \ne 0$, $f \in \mathcal{S}'$, $F(x,t) = (f * \phi_t)(x)$, $0 < q < \infty$. Then for every $a > 0$ and $N > 0$, there is a constant c depending only on ϕ, q, a and N, such that

$$|F(x,t)|^q \le c \int_0^{at} \int_{R^n} |F(y,s)|^q \left(1 + \frac{|x-y|}{s}\right)^{-qN} s^{-n} \left(\frac{s}{t}\right)^{qN} dy \frac{ds}{s}.$$

To prove the sufficiency, we will first show the following estimate:

$$\text{(2.2)} \quad M^q_{\alpha,u,\varepsilon}(f)(x) \le c \int_0^\infty \int_{R^n} |F(y,s)|^q \left(1 + \frac{|x-y|}{s}\right)^{-qN} \cdot s^{-n+\alpha q - 1} u(B_s(x))^{\varepsilon q/n}\, dy\, ds$$

for N sufficiently large. In fact, taking $a = 1$ in the mean value inequality above and observing that if $0 < s < t$, then for $\varepsilon > 0$

$$t^{\alpha q} u(B_t(x))^{\varepsilon q/n} \le c s^{\alpha q} \left(\frac{t}{s}\right)^{\alpha q + \varepsilon q \mu} u(B_s(x))^{\varepsilon q/n}$$

while for $\varepsilon \le 0$

$$t^{\alpha q} u(B_t(x))^{\varepsilon q/n} \le s^{\alpha q} (t/s)^{\alpha q} u(B_s(x))^{\varepsilon q/n},$$

we obtain

$$\{t^\alpha u(B_t(x))^{\varepsilon/n}|F(x,t)|\}^q \le c\int_0^t \int_{R^n} |F(y,s)|^q \left(1+\frac{|x-y|}{s}\right)^{-qN} \cdot s^{-n+\alpha q-1}u(B_s(x))^{\varepsilon q/n}\left(\frac{s}{t}\right)^{q(N-\tau-\alpha)}\,dy\,ds,$$

where $\tau=\varepsilon\mu$ if $\varepsilon>0$ and $\tau=0$ if $\varepsilon\le 0$. In any case, since $(s/t)^{q(N-\tau-\alpha)}\le 1$ for sufficiently large N, we obtain (2.2) by taking the sup in $t>0$.

Now, if we integrate (2.2) with respect to $u(x)\,dx$ and change the order of integration on the right, it suffices to prove that the measure

$$\sigma(y,s)\,dy\,ds = \left\{s^{-1-n+\alpha q}\int \left(1+\frac{|x-y|}{x}\right)^{-qN} u(B_s(x))^{\varepsilon q/n}u(x)\,dx\right\}dy\,ds$$

is a Carleson measure, i.e., $\sigma(B_Q)\le cv(Q)^{q/p}$. In fact, observing that

$$\left(1+\frac{|x-y|}{s}\right)^{-qN} \le \sum_{k=0}^{\infty} 2^{-kqN}\chi\left(\frac{|x-y|}{2^{k+1}s}\right),$$

where $\chi(x)$ is the characteristic function of $|x|\le 1$, we have

$$\int_Q\int_0^l \sigma(y,s)\,dy\,ds \le c\sum_{k=0}^{\infty} 2^{-kqN}\int_Q\int_0^l s^{-1-n+\alpha q} \cdot \int \chi\left(\frac{|x-y|}{2^{k+1}s}\right) u(B_s(x))^{\varepsilon q/n}u(x)\,dx\,dy\,ds.$$

Note that $x\in 2^{k+2}Q$ if $y\in Q$ and $|x-y|\le 2^{k+1}l$. Thus, performing the integration with respect to y, we see that the last sum is at most

$$c\sum_{k=0}^{\infty} 2^{-kqN+kn}\int_{2^{k+2}Q} u(x)\int_0^l s^{-1+\alpha q}u(B_s(x))^{\varepsilon q/n}\,ds\,dx. \tag{2.3}$$

In order to estimate the integral

$$I=\int_0^l s^{-1+\alpha q}u(B_s(x))^{\varepsilon q/n}\,ds, \tag{2.4}$$

note that by doubling and reverse doubling,

$$c'\left(\frac{s}{l}\right)^{n\mu} u(B_l(x)) \le u(B_s(x)) \le c\left(\frac{s}{l}\right)^{n\nu} u(B_l(x)), \qquad 0<s<l. \tag{2.5}$$

Hence, if $\alpha>0$ and $\varepsilon>-\alpha/\mu$, we have for $\varepsilon\ge 0$

$$I\le u(B_l(x))^{\varepsilon q/n}\int_0^l s^{-1+\alpha q}\,ds = cu(B_l(x))^{\varepsilon q/n}l^{\alpha q},$$

and for $\varepsilon<0$, using (2.5),

$$I\le c\int_0^l s^{-1+\alpha q}\left(\frac{s}{l}\right)^{\varepsilon q\mu} u(B_l(x))^{\varepsilon q/n}\,ds = cl^{\alpha q}u(B_l(x))^{\varepsilon q/n}.$$

On the other hand, if $\alpha\le 0$ and $\varepsilon>-\alpha/\nu$, using (2.5) we have

$$I\le c\int_0^l s^{-1+\alpha q}\left(\frac{s}{l}\right)^{\nu\varepsilon q} u(B_l(x))^{\varepsilon q/n}\,ds \le cl^{\alpha q}u(B_l(x))^{\varepsilon q/n}.$$

Therefore, (2.3) is less than or equal to

$$c\sum_{k=0}^{\infty} 2^{-kqN+kn} l^{\alpha q} \int_{2^{k+2}Q} u(B_l(x))^{\varepsilon q/n} u(x)\,dx.$$

We will show that the integral in this sum is bounded by $c2^{kM}u(Q)^{(\varepsilon q/n)+1}$, where M depends only on n, μ, ε and q. If $\varepsilon \geq 0$, the integral is at most

$$cu(c2^kQ)^{(\varepsilon q/n)+1} \leq c2^{k\eta\mu[(\varepsilon q/n)+1]} u(Q)^{(\varepsilon q/n)+1}$$

by (2.5). If, on the other hand, $\varepsilon < 0$, we use the estimate $u(B_l(x)) \geq c2^{-kn\mu}u(2^kQ)$, $x \in 2^{k+2}Q$, to majorize the integral by $c2^{-k\mu\varepsilon q}u(c2^kQ)^{(\varepsilon q/n)+1}$. By (2.5), this is less than $c2^{-k\mu\varepsilon q}2^{kn\mu[(\varepsilon q/n)+1]}u(Q)^{(\varepsilon q/n)+1}$ if $(\varepsilon q/n) + 1 \geq 0$, and less than $c2^{-k\mu\varepsilon q}u(Q)^{(\varepsilon q/n)+1}$ if $(\varepsilon q/n) + 1 < 0$. Substituting this estimate in the series above and taking N sufficiently large, we see the series converges, and it follows that $\sigma(B_Q) \leq c|Q|^{\alpha q/n}u(Q)^{(\varepsilon q/n)+1}$. By condition (1.1) we then obtain $\sigma(B_Q) \leq cv(Q)^{q/p}$, which completes the proof of the sufficiency.

As mentioned in the introduction, the sufficiency part of Theorem 1 can be modified so that for $\varepsilon \geq 0$, the assumption that $u \in D_\infty$ is not needed. We assume instead that $v \in D_\infty \cap RD_\nu$, $\alpha > -\nu\varepsilon q/p$, and of course that (1.1) holds. To see why this is so, first note that since $v \in D_\infty$ and $\varepsilon/n + 1/q \geq 0$, (1.1) for cubes is equivalent to (1.1) for balls, so that if $\varepsilon \geq 0$,

$$u(B_t(x))^{\varepsilon/n} \leq ct^{-\delta\alpha} v(B_t(x))^{\delta/p},$$

with $\delta = \varepsilon q/(n+\varepsilon q)$, $0 \leq \delta < 1$. Also, there exists $\gamma > 0$ such that $v(B_t(x)) \leq c(t/s)^{n\gamma}v(B_s(x))$ for $0 < s \leq t$, and therefore

$$t^{\alpha q}u(B_t(x))^{\varepsilon q/n} \leq ct^{\alpha(1-\delta)q}(t/s)^{n\gamma\delta q/p}v(B_s(x))^{\delta q/p}$$

for $0 < s \leq t$. Then, for N big enough, inequality (2.2) can be replaced by

$$M_{\alpha,u,\varepsilon}(f)(x) \leq c\int_0^\infty \int_{R^n} |F(y,s)|^q \left(1 + \frac{|x-y|}{s}\right)^{-qN} \cdot s^{-n+\alpha(1-\delta)q-1} v(B_s(x))^{\delta q/p}\,dy\,ds.$$

If we integrate this with respect to $u(x)dx$, it suffices to show that the function

$$\tilde{\sigma}(y,s) = s^{-1-n+\alpha(1-\delta)q}\int_{\mathbf{R}^n} \left(1 + \frac{|x-y|}{s}\right)^{-qN} v(B_s(x))^{\delta q/p}u(x)\,dx$$

induces a Carleson measure. Proceeding as usual, instead of (2.3) we get

$$\text{(2.3}'\text{)} \qquad c\sum_{k=0}^{\infty} 2^{-kqN+kn} \int_{2^{k+2}Q} u(x) \int_0^l s^{-1+\alpha q(1-\delta)} v(B_s(x))^{\delta q/p}\,ds\,dx,$$

and now in (2.4), I is

$$\int_0^l s^{-1+\alpha(1-\delta)q} v(B_s(x))^{\delta q/p}\,ds.$$

Since $\delta > 0$, reverse doubling implies that

$$I \leq cv(B_l(x))^{\delta q/p}\left(\int_0^l s^{-1+\alpha(1-\delta)q}s^{n\nu\delta q/p}\,ds\right) l^{-n\nu\delta q/p},$$

and the last integral converges for $\alpha > -\nu\varepsilon q/p$. Therefore,

$$I \le Cl^{\alpha(1-\delta)q} v(B_l(x))^{\delta q/p}.$$

Hence, $(2.3)'$ is at most

$$c\sum_{k=0}^{\infty} 2^{-kqN+kn} l^{\alpha(1-\delta)q} \int_{2^{k+2}Q} v(B_l(x))^{\delta q/p} u(x)\, dx,$$

which is less than

$$c\sum_{k=0}^{\infty} 2^{-kqN+kn} l^{\alpha(1-\delta)q} v(c2^kQ)^{\delta q/p} u(2^{k+2}Q).$$

Now applying (1.1) and then doubling for v, it follows that the last sum is less than $cv(Q)^{q/p}$ for N big enough. This completes the proof.

To prove the necessity, since u and $v \in D_\infty$, it is enough to show (1.1) for every ball. Let N be a nonnegative integer that will be determined later. Let $\phi \in C_0^\infty$ be supported in $|x| \le 1/2$, $|\phi(x)| \le 1$, and $\phi(x) = 1$ for $|x| \le 1/4$. Define

$$\lambda_\gamma = \int \phi(x) x^\gamma\, dx, \qquad |\gamma| \le N.$$

Then by Lemma 2.6 of [**4**], there is a function $\psi \in C_0^\infty$ whose support is contained in $\{1/2 \le |x| \le 1\}$ such that

$$\int_{R^n} \psi(x) x^\gamma\, dx = \lambda_\gamma, \qquad |\gamma| \le N.$$

Let $f = \phi - \psi$, and for a given ball $B = B_R(x_0)$, define $f_B(x) = f((x - x_0)/R)$. Note that $f_B(x) = 1$ if $|x - x_0| \le R/4$. Now if $\eta \in S$ with $\int \eta(x)\, dx \ne 0$ and $F_B(x,t) = (f_B * \eta_t)(x)$, we will show that

$$c\|f\|_\infty R^\alpha u(B)^{\varepsilon/n} \le M_{\alpha,u,\varepsilon}(f_B)(x) \tag{2.6}$$

for $|x - x_0| \le R/8$ and some $c > 0$ depending only on η, u, α and ε. In fact, let a be so large that $\int_{|u|>a} |\eta(u)|\, du \ne 0$ and

$$\left| \int_{|u|\le a} \eta(u)\, du \right| \ge 10\|f\|_\infty \int_{|u|>a} |\eta(u)|\, du,$$

and let $t^* = R/(8a)$. Then for $|x - x_0| \le R/8$, we have

$$\begin{aligned} |F_B(x,t^*)| &\ge \left| \int_{|x-y|\le R/8} f_B(y)\eta_{t^*}(x-y)\, dy \right| - \left| \int_{|x-y|>R/8} f_B(y)\eta_{t^*}(x-y)\, dy \right| \\ &\ge \left| \int_{|u|\le a} \eta(u)\, du \right| - \|f\|_\infty \int_{|u|>a} |\eta(u)|\, du \ge c\|f\|_\infty. \end{aligned}$$

Therefore, by doubling, we get (2.6).

On the other hand, since f_B is supported in B $(= B_R(x_0))$ and its moments of order $\le N$ vanish, a standard computation gives

$$N(f_B)(x) \le c\|f\|_\infty \left(1 + \frac{|x - x_0|}{R}\right)^{-N-1-n}, \qquad x \in R^n.$$

Therefore, by choosing N large enough and using the fact that v is doubling, we obtain $\|f_B\|_{H^p_v} \leq c\|f\|_\infty v(B)^{1/p}$; in fact,

$$\begin{aligned}\|f_B\|^p_{H^p_v} &= \int N(f_B)(x)^p v(x)\,dx \\ &\leq c\|f\|^p_\infty \sum_{k=0}^{\infty}(2^k)^{-N-1-n}\int_{|x-x_0|\leq 2^k R} v(x)\,dx \\ &\leq c\|f\|^p_\infty \sum_{k=0}^{\infty}(2^k)^{-N-1-n+n\sigma} v(B),\end{aligned}$$

where σ is the doubling order of v, and we have only to choose N so large that the sum converges.

Finally, if we take the qth power of (2.6), integrate over $B_{R/8}(x_0)$ with respect to $u(x)\,dx$ and use the norm inequality and the fact that $u \in D_\infty$, (1.1) follows.

3. Proofs of Theorems 2, 3 and 4. The proofs are an application of Theorem 1 and the following lemma, which is a generalization of two inequalities, one given by Welland in [**11**] and the other by Hedberg in [**7**].

LEMMA (3.1). (a) *If N_0 is the radial maximal function defined with $\phi(x) = e^{-|x|^2}$, $\alpha > 0$ and $u \in D_\mu$ with $0 < \varepsilon < \alpha/\mu$, then*

$$N_0(I_\alpha f)(x) \leq c\{M_{\alpha,u,\varepsilon}(f)(x) M_{\alpha,u,-\varepsilon}(f)(x)\}^{1/2}$$

for every $f \in S_\alpha$.

(b) *If $\alpha, q, \varepsilon > 0$, $u \in D_\infty$, $v \in D_\infty$ and the pair (u, v) satisfies* (1.2), *then*

$$|I_\alpha f(x)| \leq c\|f\|^\Theta_{H^p_v}[M_{\alpha,u,-\varepsilon}(f)(x)]^{1-\Theta}$$

for every $f \in S_\alpha$, where $\Theta = \varepsilon/(\varepsilon + (n/q))$.

PROOF. If $f \in S_\alpha$, we have the following representation formula for I_α:

$$I_\alpha f(x) = c\int_0^\infty t^{\alpha-1}F(x,t)\,dt, \tag{3.2}$$

where $F(x,t)$ is the convolution of f with a fixed approximation to the identity ϕ_t, $\phi \in S$, ϕ radial. In fact, since $|F(x,t)| \leq ct^{-n-N_\alpha-1}$ for $t \geq 1$, and $|F(x,t)| \leq c$ for $t < 1$, the integral above is absolutely convergent and (3.2) follows by taking Fourier transforms.

To prove (a), observe that when $\phi(x) = e^{-|x|^2}$ the function $G(x,t) = (f * \phi_{t^{1/2}})(x)$ satisfies the heat equation, and

$$(f * \phi_{t^{1/2}} * \phi_{s^{1/2}})(x) = (f * \phi_{(t+s)^{1/2}})(x).$$

Consequently, by changing variables in (3.2), we have

$$\begin{aligned}(I_\alpha f * \phi_{s^{1/2}})(x) &= \frac{1}{2}\int_0^\infty t^{(\alpha/2)-1}G(x,t+s)\,dt \\ &= \frac{1}{2}\int_s^{s+\delta}(t-s)^{(\alpha/2)-1}G(x,t)\,dt + \frac{1}{2}\int_{s+\delta}^\infty (t-s)^{(\alpha/2)-1}G(x,t)\,dt \\ &= \frac{1}{2}I_1 + \frac{1}{2}I_2 \quad \text{for any } s,\delta > 0.\end{aligned}$$

For I_2, write

$$\begin{aligned}
|I_2| &\leq \int_{s+\delta}^{\infty}(t-s)^{(\alpha/2)-1}u\left(B_{(t-s)^{1/2}}(x)\right)^{-\varepsilon/n}u\left(B_{(t-s)^{1/2}}(x)\right)^{\varepsilon/n}|F(x,t^{1/2})|\,dt \\
&\leq \int_{s+\delta}^{\infty}(t-s)^{-1}u\left(B_{(t-s)^{1/2}}(x)\right)^{-\varepsilon/n}t^{\alpha/2}u\left(B_{t^{1/2}}(x)\right)^{\varepsilon/n}|F(x,t^{1/2})|\,dt \\
&\leq 2M_{\alpha,u,\varepsilon}(f)(x)\int_{\delta^{1/2}}^{\infty}t^{-1}u(B_t(x))^{-\varepsilon/n}\,dt \\
&= 2M_{\alpha,u,\varepsilon}(f)(x)J_2(\delta).
\end{aligned}$$

For I_1, since $u \in D_\mu$ and $\varepsilon < \alpha/\mu$, we have

$$(t-s)^{\alpha/2}u\left(B_{(t-s)^{1/2}}(x)\right)^{-\varepsilon/n} \leq ct^{\alpha/2}u\left(B_{t^{1/2}}(x)\right)^{-\varepsilon/n},$$

$t > s$, and therefore as above,

$$\begin{aligned}
|I_1| &\leq cM_{\alpha,u,-\varepsilon}(f)(x)\cdot 2\int_0^{\delta^{1/2}}t^{-1}u(B_t(x))^{\varepsilon/n}\,dt \\
&= 2cM_{\alpha,u,-\varepsilon}(f)(x)J_1(\delta).
\end{aligned}$$

Consequently,

$$N_0(I_\alpha f)(x) \leq 2cM_{\alpha,u,-\varepsilon}(f)(x)J_1(\delta) + 2M_{\alpha,u,\varepsilon}(f)(x)J_2(\delta)$$

for every $\delta > 0$. Now, we minimize the last expression in δ and find that the value δ_0 at which the minimum is achieved satisfies

$$u\left(B_{\delta_0^{1/2}}(x)\right)^{2\varepsilon/n} = \frac{M_{\alpha,u,\varepsilon}(f)(x)}{cM_{\alpha,u,-\varepsilon}(f)(x)}.$$

Also, by the sort of argument used to estimate (2.4), we have that

$$J_1(\delta) \leq cu(B_{\delta^{1/2}}(x))^{\varepsilon/n} \quad \text{and} \quad J_2(\delta) \leq cu(B_{\delta^{1/2}}(x))^{-\varepsilon/n}.$$

Hence, we get part (a) of the lemma.

To prove (b), write

$$I_\alpha f(x) = c\left(\int_0^\delta + \int_\delta^\infty\right)t^{\alpha-1}F(x,t)\,dt = \tilde{I}_1 + \tilde{I}_2.$$

Since $|F(x,t)| \leq N(f)(y)$ if $|x-y| < t$, we obtain by integration that $|F(x,t)| \leq \|f\|_{H_v^p}v(B_t(x))^{-1/p}$. Thus,

$$\left|\tilde{I}_2\right| \leq c\|f\|_{H_v^p}\int_\delta^\infty t^{\alpha-1}v(B_t(x))^{-1/p}\,dt.$$

Now since (u,v) satisfies (1.2),

$$\left|\tilde{I}_2\right| \leq c\|f\|_{H_v^p}\int_\delta^\infty t^{-1}u(B_t(x))^{-1/q}\,dt = c\|f\|_{H_v^p}\tilde{J}_2(\delta).$$

For $\tilde{I}_1$, we proceed as in the estimate of I_1 in part (a) and we get

$$\begin{aligned}
\left|\tilde{I}_1\right| &\leq cM_{\alpha,u,-\varepsilon}(f)(x)\int_0^\delta t^{-1}u(B_t(x))^{\varepsilon/n}\,dt \\
&= cM_{\alpha,u,-\varepsilon}(f)(x)\tilde{J}_1(\delta).
\end{aligned}$$

Now the proof follows as in part (a).

PROOF OF THEOREM 2. To prove the sufficiency, let $1/q_1 = 1/q - \varepsilon/n$ and $1/q_2 = 1/q + \varepsilon/n$ for $\varepsilon > 0$ to be chosen small. Note that $q/2q_1 + q/2q_2 = 1$. Thus by Lemma (3.1)(a) and Hölder's inequality, we have for $f \in \mathcal{S}_\alpha$ and $\varepsilon < \alpha/\mu$

$$\int N_0(I_\alpha f)(x)^q u(x)\,dx \le c \int M_{\alpha,u,\varepsilon}(f)(x)^{q/2} M_{\alpha,u,-\varepsilon}(f)(x)^{q/2} u(x)\,dx$$
$$\le c\|M_{\alpha,u,\varepsilon}(f)\|_{L_u^{q_1}}^{q/2} \|M_{\alpha,u,-\varepsilon}(f)\|_{L_u^{q_2}}^{q/2}.$$

Observe that (1.2) is equivalent to two versions of (1.1), one for q_1, p, ε and the other for q_2, p, $-\varepsilon$. Since $q > p$, by choosing ε small enough, we have $q_1 \ge p$, $q_2 \ge p$. Hence the desired result follows by applying Theorem 1.

To prove the necessity, let I_0 be the unit cube centered at the origin and choose $f_0 \in \mathcal{S}_\alpha$, $|f_0(x)| \le 1$, $\operatorname{supp}(f_0) \subset I_0$, such that $I_\alpha f_0$ is not identically zero in I_0. We will consider $\alpha = n + 2l$, $l = 0, 1, 2, \ldots$, since for other α's the proof is analogous. Since $I_\alpha f_0$ is continuous, there is a cube $\tilde{I}_0 \subset I_0$ such that $|I_\alpha f_0(x)| > c_0 > 0$ for $x \in \tilde{I}_0$. If I is any cube with center x_I and edgelength δ, let L_I be defined by $L_I(x) = x_I + \delta x$. Then $L_I(I_0) = I$. We take $f_I(x) = f_0(L_I^{-1}(x))$ and have

$$\begin{aligned} I_\alpha f_I(x) &= \int f_I(y)|x-y|^{\alpha-n} \log|x-y|\,dy \\ &= \delta^n \int f_0(z)|x - L_I(z)|^{2l} \log|x - L_I(z)|\,dz \\ &= \delta^\alpha I_\alpha f_0(t) \quad \text{if } x = x_I + \delta t,\ t \in \tilde{I}_0, \end{aligned}$$

by using the moment properties of f_0. Hence, $|I_\alpha f_I(x)| \ge c_0|I|^{\alpha/n}$ for $x \in \tilde{I} = L_I(\tilde{I}_0)$. Thus, since $|\tilde{I}| \ge c|I|$, $c > 0$, and $u \in D_\infty$, $\|I_\alpha f_I\|_{H_u^q} \ge c_0|I|^{\alpha/n} u(I)^{1/q}$. On the other hand, as before, $\|f_I\|_{H_v^p} \le cv(I)^{1/p}$, and the result follows.

PROOF OF THEOREM 3. Since $v \in RH_{p^*/p}$, we have $v \in D_\infty$, $v^*(x) = [v(x)]^{p^*/p} \in D_\infty$ and $|Q|^{\alpha/n} v^*(Q)^{1/p^*} \le cv(Q)^{1/p}$ for every cube Q. Therefore, from Lemma (3.1)(b) with $u = v^*$ and $q = p^*$,

$$|I_\alpha f(x)| \le c\|f\|_{H_v^p}^{\Theta} \left[M_{\alpha,v^*,-\varepsilon}(f)(x)\right]^{1-\Theta},$$

where $\varepsilon > 0$ will be chosen small and $\Theta = \varepsilon/(\varepsilon + (n/p^*))$. Raising this inequality to the power q and integrating with respect to $u(x)\,dx$, we obtain

$$\|I_\alpha f\|_{L_u^q} \le c\|f\|_{H_v^p}^{\Theta} \left(\int_{R^n} [M_{\alpha,v^*,-\varepsilon}(f)(x)]^{(1-\Theta)q} u(x)\,dx\right)^{1/q}.$$

To prove the theorem, it is enough to show that

$$\left(\int_{R^n} [M_{\alpha,v^*,-\varepsilon}(f)(x)]^{(1-\Theta)q} u(x)\,dx\right)^{1/q} \le c\|f\|_{H_v^p}^{1-\Theta}. \tag{3.3}$$

Since the proof of (3.3) is similar to the proof of the sufficiency in Theorem 1, we will omit some of the details. In fact, it is clear that (2.2) is true with v^*, $-\varepsilon$ and $(1-\Theta)q$ instead of u, ε and q, respectively. Integrating (2.2) with respect to $u(x)\,dx$ and changing the order of integration on the right it is enough to show that the measure

$$\tilde{\sigma}(y,s)\,dy\,ds = \left\{ s^{-1-n+\alpha(1-\Theta)q} \int \left(1+\frac{|x-y|}{s}\right)^{-(1-\Theta)qN} \cdot v^*(B_s(x))^{-\varepsilon(1-\Theta)q/n} u(x)\,dx \right\} dy\,ds$$

is a Carleson measure, i.e., $\tilde{\sigma}(B_Q) \le cv(Q)^{(1-\Theta)q/p}$. Proceeding as in §2, we can show that

$$\tilde{\sigma}(B_Q) \le c\sum_{k=0}^{\infty} 2^{-k(1-\Theta)qN+kn} l^{\alpha(1-\Theta)q} \int_{2^{k+2}Q} v^*(B_l(x))^{-\varepsilon(1-\Theta)q/n} u(x)\,dx.$$

Since $v^* \in D_\infty$, $v^* \in D_{\overline{\mu}}$ for some $\overline{\mu}$, and then

$$\tilde{\sigma}(B_Q) \le c\sum_{k=0}^{\infty} 2^{-k(1-\Theta)qN+kn-k\varepsilon(1-\Theta)q\overline{\mu}} l^{\alpha(1-\Theta)q} v^*(Q)^{-\varepsilon(1-\Theta)q/n} u(2^{k+2}Q).$$

Also $v \in D_{\overline{\nu}}$ for some $\overline{\nu}$, and the pair (u,v) satisfies (1.2). Consequently,

$$\begin{aligned} u(2^{k+2}Q) &\le c|2^{k+2}Q|^{-\alpha q/n} v(2^{k+2}Q)^{q/p} \\ &\le c2^{-k\alpha q+k\overline{\nu}nq/p}|Q|^{-\alpha q/n} v(Q)^{q/p}. \end{aligned}$$

Hence, by choosing N big enough, we obtain

$$\tilde{\sigma}(B_Q) \le c|Q|^{-\alpha\Theta q/n} v^*(Q)^{-\varepsilon(1-\Theta)q/n} v(Q)^{q/p}.$$

Now, $v^*(Q) \le c|Q|^{-\alpha p^*/n} v(Q)^{p^*/p}$, and observing that $p^*\varepsilon(1-\Theta)q/(pn) = \Theta q/p$ and $\alpha p^*\varepsilon(1-\Theta)q/n = \alpha\Theta q$, we get

$$\tilde{\sigma}(B_Q) \le cv(Q)^{(1-\Theta)q/p}.$$

Finally, as $q>p$, for ε sufficiently small we have $(1-\Theta)q \ge p$, and by Lemma (2.1) we have (3.3).

PROOF OF THEOREM 4. We shall use the following lemma from [**10**].

LEMMA (3.4). *Let $u \in D_\infty$, $\{a_k\}_{k=1}^m$ be a sequence of distinct points in R^n, $0 \le \beta_k \le \beta$, $k=1,\dots,m$, and*

$$\Pi_\beta(x) = (1+|x|)^\beta \prod_{k=1}^m \left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\beta_k}.$$

Then there exists a constant $c>0$ such that

$$\int_Q u(x)\Pi_\beta(x)\,dx \ge c|Q|^{\beta/n} \int_Q u(x)\,dx$$

for every cube Q.

To prove the theorem, take δ such that $|\delta|$ is small and define $M_\delta = M_{\alpha,1,\delta}$, $M_{-\delta} = M_{\alpha,1,-\delta}$ and

$$k_\delta(x) = (1+|x|)^{\delta p} \prod_{\alpha_k \ne 0} \left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{\delta p}.$$

Then by Lemma (3.1)(a) and Hölder's inequality, we have

$$\|I_\alpha f\|_{H_u^p}^p \leq c\left(\int_{R^n} M_\delta f(x)^p k_\delta^{-1}(x)u(x)\,dx\right)^{1/2}\left(\int_{R^n} M_{-\delta} f(x)^p k_\delta(x)u(x)\,dx\right)^{1/2}.$$

Note that $k_\delta u$, $k_\delta^{-1}u \in D_\infty$ (see [**10**] for a proof), $k_\delta^{-1}(x) = k_{-\delta}(x)$ and $M_{\pm\delta} = M_{\alpha\pm\delta, k_{\mp\delta}u,0}$. Thus, by Theorem 1, it is enough to show that

$$|Q|^{(\alpha\pm\delta)/n}\left(\int_Q k_{\mp\delta}(x)u(x)\,dx\right)^{1/p} \leq c\left(\int_Q v(x)\,dx\right)^{1/p}.$$

In fact, if we take $\beta = (\alpha\pm\delta)p$, $\beta_k = (\alpha_k \pm \delta)p$ and apply Lemma (3.4), observing that

$$k_{\pm\delta}(x)\Pi_\alpha(x)^p = \Pi_\beta(x) \quad\text{and}\quad v(x) = \Pi_\alpha(x)^p u(x) = \Pi_\beta(x)k_{\mp\delta}(x)u(x),$$

we get the last inequality. This completes the proof of Theorem 4.

4. Applications. In this section, we will show the following corollary of Theorems 2 and 4.

COROLLARY. *Let $1 < p \leq q < \infty$, $u, v \in D_\infty$, v satisfy conditions* (i)–(v) *below and if $p < q$*

$$|Q|^{1/n}u(Q)^{1/q} \leq cv(Q)^{1/p} \quad \textit{for every cube } Q,$$

or if $p = q$, u and v satisfy the hypothesis of Theorem 4 with $\alpha = 1$. Then

$$\|f\|_{L_u^q} \leq c\|\nabla f\|_{L_v^p} \quad \textit{for every } f \in S_{0,0},$$

where ∇f denotes the gradient of f.

We recall the class of weights introduced by Adams in [**2**]. For a nonnegative integer k, $\mathbf{P}_k$ denotes the set of polynomials of degree less than or equal to k and $\mathbf{P}_{-1} = \{0\}$. If $\{p_j\}_{j=1}^J \subset R^n$, $m_0 \in Z$, $m_j \in Z$, $m_j \geq 0$, $j = 1, \ldots, J$, we denote

$$\mathbf{R}_k = \{R \in \mathbf{P}_k\colon \text{ for each } j = 1, \ldots, J,\ D^\gamma R(p_j) = 0 \text{ if } |\gamma| < m_j\}$$

and $M = [\sum_{j=1}^J m_j] - 1$, $N = M + m_0$. The weights in [**2**] have the form $v(x) = Q(x)^p w(x)$, $1 < p < \infty$, where

(i) $Q(x) = (1+|x|)^{m_0}\prod_{j=1}^J |x - p_j|^{m_j}$, $\sum_{j=0}^J m_j \geq 0$,
(ii) $\mathbf{P}_N \cup \mathbf{R}_M$ spans $\mathbf{P}_M$,
(iii) $w \in A_p$,
(iv) $w(x)(1+|x|)^{p(1-n)} \in A_p$ if $N \geq 0$,
(v) if $m_j \neq 0$, $w(x)|x - p_j|^{p(n-1)} \in A_p$, $j = 1, \ldots, J$.

For such v and $f \in S_{0,0}$, it is shown in [**2**] that $\|f\|_{H_v^p} \approx \|f\|_{L_v^p}$ (the distribution action of an $f \in S_{0,0}$ is defined as usual to be $\langle f, \phi\rangle = \int f\phi\,dx$). We remark that when $n = 1$, we have $\mathbf{R}_M = \{0\}$, and the conditions above amount simply to assuming that $v(x) = |Q(x)|^p w(x)$, where $Q(x)$ is any polynomial and $w \in A_p$.

To prove the corollary, we write

$$f(x) = I_1\left(\sum_{j=1}^n R_j\left(\frac{\partial f}{\partial x_j}\right)\right)(x),$$

where R_j are the Riesz transforms and $f \in S_{0,0}$. Let u and v satisfy the hypothesis of the corollary. Then by either Theorem 2 if $p < q$ or Theorem 4 if $p = q$, we have

$$\|f\|_{L^q_u} \leq c \sum_{j=1}^{n} \left\| R_j \left(\frac{\partial f}{\partial x_j} \right) \right\|_{H^p_v}.$$

Now since $R_j \partial f/\partial x_j \in S_{0,0}$, we have

$$\left\| R_j \left(\frac{\partial f}{\partial x_j} \right) \right\|_{H^p_v} \approx \left\| R_j \left(\frac{\partial f}{\partial x_j} \right) \right\|_{L^p_v},$$

and since $\partial f/\partial x_j \in S_{0,0}$, $\|R_j(\partial f/\partial x_j)\|_{L^p_v} \leq c\|\partial f/\partial x_j\|_{L^p_v}$ by [**1**]. This completes the proof of the corollary.

Now we prove the Sobolev type inequality announced in the introduction. In fact, to prove the necessity of (1.4) and (1.5) for (1.3), take any function $\phi \in C_0^\infty$, $\phi \not\equiv 0$, whose support is disjoint from the origin. Then by estimating the norms of $f(x) = \phi(\lambda x)$ in (1.3) and letting $\lambda \to 0$, $\lambda \to \infty$, (1.4) follows. To see (1.5), take $\phi_R \in C_0^\infty$ such that $\operatorname{supp} \phi_R \subset \{R \leq |x| \leq R+1\}$ and $\operatorname{supp} \phi_R$ has diameter at most 1. Then, by taking $f = \phi_R$ in (1.3) and letting $R \to \infty$, we have $\gamma \leq \beta$, which together with (1.4) implies (1.5).

To prove that (1.4) and (1.5) imply (1.3), we assume first that $1/q > -\gamma/n$. In this case, we will prove (1.3) for any $f \in C_0^\infty$. Let $u(x) = |x|^{q\gamma}$ and $v(x) = |x|^{p\beta}$. Then u and v are locally integrable and satisfy (1.2) if $p < q$ or the hypothesis of Theorem 4 with $\alpha = 1$ if $p = q$. Also, it is easy to see that v satisfies (i)–(v) above if $\beta + (n/p) > 0$ and $\beta + (n/p) \neq n + m$, $m = 0, 1, 2, \ldots$. Therefore, by the corollary, (1.3) follows for every $f \in S_{0,0}$ provided that $\beta + (n/p) > 0$ and $\beta + (n/p) \neq n+m$, $m = 0, 1, 2, \ldots$. To see that it holds for every $f \in C_0^\infty$ and that the restrictions on β can be dropped, we will use the density lemma below together with a change of variables.

LEMMA (4.1). *Let $1 < p < \infty$, $f \in C_0^\infty$, N be an integer ≥ -1 and $\int f(x) x^\sigma \, dx = 0$ for $|\sigma| \leq N$ (if $N = -1$ no moment conditions are imposed). Let $v(x)$ be a weight satisfying* (i)–(v) *above with $N = [\sum_{j=0}^{J} m_j] - 1$. Then there exists a sequence $\{f_k\} \in S_{0,0}$ such that $f_k \to f$ and $\partial f_k/\partial x_j \to \partial f/\partial x_j$, $j = 1, \ldots, n$, in L^p_v and L^2.*

PROOF. We take $\hat\eta \in S$ with $\hat\eta(x) = 1$ for $|x| \leq 1$ and $\hat\eta(x) = 0$ for $|x| > 2$, and define $\psi_k(x) = \eta_{1/k}(x) - \eta_k(x)$ and $f_k(x) = (f * \psi_k)(x)$, $k = 1, 2, \ldots$. Since $\hat\psi_k(x) = \hat\eta(x/k) - \hat\eta(kx)$, $\hat\psi_k$ has support contained in $1/k \leq |x| \leq 2k$, and therefore, $f_k \in S_{0,0}$. Clearly, $f_k \to f$ in L^2 as $k \to \infty$. Furthermore, $f_k \to f$ pointwise since $f * \eta_{1/k} \to f$ and $f * \eta_k \to 0$ pointwise. If $N_0(f)$ is formed by using η as the convolver, then $|f_k(x)| \leq 2N_0(f)(x) \leq c(1+|x|)^{-n-N-1}$. Moreover, $v(x) = Q(x)^p w(x) \leq c(1+|x|)^{(N+1)p} w(x)$, and since any $w \in A_p$ satisfies

$$\int_{R^n} \frac{w(x)}{(1+|x|)^{np}} \, dx < \infty,$$

it follows that $N_0(f) \in L^p_v$. Thus, by the Lebesgue dominated convergence theorem, $f_k \to f$ in L^p_v. The argument for the derivatives is similar.

We now return to the proof of (1.3). As before, we assume $1/q + \gamma/n > 0$ and that (1.4) and (1.5) hold, and let $u(x) = |x|^{q\gamma}$ and $v(x) = |x|^{p\beta}$. In case $n > 1$, we will apply Lemma (4.1) with $m_j = 0$ for $j = 0, 1, \ldots, J$ (so that $N = -1$ and $Q \equiv 1$). The hypothesis of the lemma requires in this case that $v \in A_p$, which amounts to $0 < \beta + n/p < n$. We already know that (1.3) holds for $f \in \mathcal{S}_{0,0}$ for this range. Note that (1.4) together with $1/q + \gamma/n > 0$ requires that $1 < \beta + n/p$. Thus, when $n > 1$, we obtain (1.3) for $f \in C_0^\infty$ and $1 < \beta + n/p < n$ by passing to the limit and applying Lemma (4.1).

If $n = 1$, note that for any $f \in C_0^\infty$, f' has integral 0. Take $N = 0$, $m_0 = 0$, $J = 1$ and $m_1 = 1$ in Lemma (4.1). Then $Q(x) = |x|$ and v may be written $v = Q^p w$ with $w(x) = |x|^{\beta p - p}$. The conditions on w required by Lemma (4.1) then amount to $p - 1 < \beta p < 2p - 1$. Applying the lemma to f' shows that there exists $f_k \in \mathcal{S}_{0,0}$ such that $f_k \to f'$ in $L^p_{|x|^{\beta p}}$ and L^2, $p - 1 < \beta p < 2p - 1$. Now write $g_k(x) = \int_{-\infty}^x f_k(t)\,dt$ and note that since $\hat{g}_k(x) = \hat{f}_k(x)/x$, we have $g_k \in \mathcal{S}_{0,0}$. Of course, $g_k' \to f'$ in $L^p_{|x|^{p\beta}}$ and L^2. We will show that $g_k \to f$ pointwise. Fix x and choose $a < 0$ with $a < x$. Then

$$\begin{aligned}
|g_k(x) - f(x)| &\le \int_{-\infty}^x |f_k(t) - f'(t)|\,dt \\
&= \left(\int_{-\infty}^a + \int_a^x\right) |f_k(t) - f'(t)|\,dt \\
&\le \|f_k - f'\|_{L^p_{|x|^{p\beta}}} \left(\int_{-\infty}^a |t|^{-p'\beta}\,dt\right)^{1/p'} + \|f_k - f'\|_{L^2} |x - a|^{1/2},
\end{aligned}$$

where $1/p + 1/p' = 1$. If $\beta p' > 1$, we obtain $g_k \to f$. Now (1.3) follows for every $f \in C_0^\infty$ and $1 < \beta + 1/p < 2$ by first applying (1.3) to g_k for such β and then passing to the limit.

At this point we have shown for fixed p and q that if $\gamma > -n/q$ and (1.4) and (1.5) hold, then (1.3) holds for $f \in C_0^\infty$ provided $1 < \beta + n/p < n$ if $n > 1$ or $1 < \beta + 1/p < 2$ if $n = 1$. Next, we claim that if (1.3) holds for $f \in C_0^\infty$ with support disjoint from the origin with p, q and a pair of exponents γ_0, β_0, then it also holds for such f with p, q and the exponents γ, β given by $\gamma = \gamma_0(1 + \alpha) + \alpha n/q$, $\beta = \beta_0(1 + \alpha) + \alpha(n/p - 1)$ for every $\alpha \neq -1$. It is not hard to see that this fact, when combined with the remarks above, implies that (1.3) holds for $f \in C_0^\infty$ with support disjoint from the origin provided that (1.4) and (1.5) hold, without any further restrictions on γ and β. Note that $\alpha = -1$ corresponds to $1/q = -\gamma/n$.

To show the claim, consider the change of variables $y = \phi_\alpha(x) = |x|^\alpha x$, $\alpha \neq -1$. We shall use the notations ϕ_α' and $\|\phi_\alpha'\|$ respectively for the Jacobian matrix and the absolute value of the Jacobian of ϕ_α. It is easy to see that $\|\phi_\alpha'(x)\| \le c|x|^{\alpha n}$. For $x \neq 0$, the inverse of ϕ_α is $x = \phi_\alpha^{-1}(y) = |y|^{-\alpha/(\alpha+1)} y$. Since $|x| = |y|^{1/(\alpha+1)}$ and $\|(\phi_\alpha^{-1})'\| \le c|y|^{-\alpha n/(\alpha+1)}$, we have by changing variables

$$\|f\|_{L^q_{|x|^{q\gamma}}} \le c \left(\int_{R^n} |(f \circ \phi_\alpha^{-1})(y)|^q |y|^{(q\gamma - \alpha n)/(\alpha+1)}\,dy\right)^{1/q}.$$

Therefore, since $(q\gamma - \alpha n)/(\alpha + 1) = q\gamma_0$ and $f \circ \phi_\alpha^{-1} \in C_0^\infty$ if $f \in C_0^\infty$ with support disjoint from the origin, we obtain from (1.3) that

$$\|f\|_{L^q_{|x|^{q\gamma}}} \le c \left(\int_{R^n} |\nabla(f \circ \phi_\alpha^{-1})(y)|^p |y|^{p\beta_0}\,dy\right)^{1/p}$$

for such f. Now observe that $\nabla(f\circ\phi_\alpha^{-1})(y)=(\phi_\alpha^{-1})'(y)(\nabla f)(\phi_\alpha^{-1}(y))$, so that

$$|\nabla(f\circ\phi_\alpha^{-1})(y)|^p\le c|y|^{-\alpha p/(\alpha+1)}|(\nabla f)(\phi_\alpha^{-1}(y))|^p.$$

Consequently, by changing variables again, we obtain

$$\|f\|_{L^q_{|x|^{q\gamma}}}\le c\left(\int_{R^n}|\nabla f(x)|^p|x|^{(\alpha+1)p\beta_0-\alpha p+\alpha n}\,dx\right)^{1/p}.$$

This proves the claim.

Finally, we must show that in case $1/q>-\gamma/n$, (1.4) and (1.5) imply (1.3) for any $f\in C_0^\infty$. We know this to be true if the support of f is disjoint from 0. Choose $\psi_k\in C_0^\infty$ with $\psi_k(x)=0$ if $|x|\le 1/k$, $\psi_k(x)=1$ if $|x|\ge 2/k$, $|\psi_k(x)|\le 1$, and $|\nabla\psi_k(x)|\le k$. For $f\in C_0^\infty$, we write $\|f\|_{L^q_{|x|^{q\gamma}}}\le\|f\psi\|_{L^q_{|x|^{q\gamma}}}+\|f(1-\psi_k)\|_{L^q_{|x|^{q\gamma}}}$. Since $f\psi_k$ has support disjoint from 0, (1.3) can be applied to the first term on the right, giving

$$\begin{aligned}\|f\|_{L^q_{|x|^{q\gamma}}}&\le c\|\nabla(f\psi_k)\|_{L^p_{|x|^{p\beta}}}+\|f(1-\psi_k)\|_{L^q_{|x|^{q\gamma}}}\\&\le c\|\psi_k\nabla f\|_{L^p_{|x|^{p\beta}}}+c\|f\nabla\psi_k\|_{L^p_{|x|^{p\beta}}}+\|f(1-\psi_k)\|_{L^q_{|x|^{q\gamma}}}.\end{aligned}$$

The fact that $1/q>-\gamma/n$ implies that $|x|^{q\gamma}$ is locally integrable, and therefore, the third term on the right tends to 0 as $k\to\infty$. Moreover, for the second term, we have

$$\|f\nabla\psi_k\|_{L^p_{|x|^{p\beta}}}\le k\left(\int_{1/k\le|x|\le 2k}|f|^p|x|^{p\beta}\,dx\right)^{1/p}\le ck^{1-\beta-n/p}\|f\|_\infty,$$

which also tends to 0 since $\beta>1-n/p$ due to (1.4) and the condition $1/q>-\gamma/n$. This completes the proof of (1.3).

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